

§2.3 Differential Formulas

• Basic Formulas: Notation: $f'(x) = \frac{df}{dx}$.

★ Power function: For any n , $(x^n)' = n \cdot x^{n-1}$. In particular, $(c)' = 0$, $(x)' = 1$.

eg. 1. $(\sqrt[3]{x})' = (x^{\frac{1}{3}})' \stackrel{n=\frac{1}{3}}{=} \frac{1}{3} \cdot x^{\frac{1}{3}-1} = \frac{1}{3} \cdot x^{-\frac{2}{3}}$

$(\frac{1}{x})' = (x^{-1})' \stackrel{n=-1}{=} (-1) \cdot x^{-1-1} = (-1) \cdot x^{-2} = \frac{-1}{x^2}$

• Derivative Rules:

Sum/Difference: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$.

Constant multiple: $(c \cdot f(x))' = c \cdot (f(x))'$ for any constant c .

★ Product: $[f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$ or $(f \cdot g)' = f'g + f \cdot g'$

★ Quotient: $[\frac{f(x)}{g(x)}]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ or $(\frac{f}{g})' = \frac{f'g - f \cdot g'}{g^2}$

eg. 2. Let $f(x) = 5 - \frac{3}{\sqrt{x}} + 2x^{3.5}$. Compute $f'(x)$.

Solution: $f'(x) = 5' - (\frac{3}{\sqrt{x}})' + (2 \cdot x^{3.5})'$
 $= 0 - 3 \cdot (x^{-\frac{1}{2}})' + 2 \cdot (x^{3.5})'$ } Hint: $\frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$
 $= 0 - 3 \cdot (-\frac{1}{2}) \cdot x^{-\frac{1}{2}-1} + 2 \cdot 3.5 \cdot x^{3.5-1} = \frac{3}{2} \cdot x^{-\frac{3}{2}} + 7 \cdot x^{2.5}$

eg. 3. Let $g(x) = (\frac{1}{x^5} - 2x)(\frac{1}{\sqrt{x}} + \pi)$. Compute $g'(x)$.
(5/16)

Solution: $g'(x) = [\frac{1}{x^5} - 2x]'(\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x)(\frac{1}{\sqrt{x}} + \pi)'$
 $= (x^{-5} - 2x)'(\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x)(x^{-\frac{1}{2}} + \pi)'$
 $= (-5x^{-6} - 2) \cdot (\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x) \cdot (-\frac{1}{2} \cdot x^{-\frac{3}{2}} + 0)$

eg 4. Let $f(x) = \frac{2}{x+1}$. Find the tangent line of $f(x)$ at $(3, \frac{1}{2})$.

Recall, the slope of the tangent line at $x=3$ equals the derivative at $x=3$.

ie. We need to compute $f'(3)$ (or $\frac{df}{dx}|_{x=3}$)

$$\begin{aligned} \text{Solution: } f'(x) &= \frac{2'(x+1) - 2(x+1)'}{(x+1)^2} \quad (\text{quotient rule}) \quad 2' = 0, (x+1)' = x' + 1' = 1 + 0 \\ &= \frac{0 - 2}{(x+1)^2} = \frac{-2}{(x+1)^2} \Rightarrow f'(3) = \frac{-2}{(3+1)^2} = \frac{-2}{16} = -\frac{1}{8} \end{aligned}$$

Therefore, the tangent formula:

$$y - \frac{1}{2} = \left(-\frac{1}{8}\right) \cdot (x - 3)$$

Higher order derivatives:

The derivative function of $f'(x)$ is called the second derivative of $f(x)$, denoted

$$f''(x) = (f'(x))' = \frac{d^2 f}{dx^2} = \frac{d}{dx} \left(\frac{df}{dx} \right)$$

In the same way, we can define third/fourth ... order derivative as $f'''(x) = [f''(x)]'$, ...

n -th order derivative is also denoted as $f^{(n)}(x)$, eg. $f^{(4)}(x) = f''''(x) = [f'''(x)]'$

Three important physical functions: Displacement $s(t)$, Velocity $v(t)$, Acceleration $a(t)$.

$$v(t) = s'(t), \quad a(t) = v'(t) \quad \text{ie. } a(t) \text{ is the second order derivative of } s(t), \text{ ie. } a(t) = s''(t).$$

eg 5. A particle moves according $s(t) = t^3 - 6t^2 + 5$, $t \geq 0$.

(Fib). (a) Find the velocity at time t .

(b) What is the acceleration after 6 seconds?

$$\text{Solution: (a) } v(t) = s'(t) = (t^3 - 6t^2 + 5)' = 3t^2 - 6 \cdot 2t + 0 = 3t^2 - 12t$$

$$(b) a(t) = (v(t))' = (3t^2 - 12t)' = 3 \cdot 2t - 12 = 6t - 12$$

$$\text{so } a(6) = (6t - 12)|_{t=6} = 6 \cdot 6 - 12 = 24 \quad \text{ft/s}^2$$

§2.4 Trigonometric Derivatives

- Key formulas: • $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$, $(\sec x)' = \tan x \sec x$
- $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.

eg1. Let $f(x) = 2\sin x - x \cdot \cos x$. Find $f'(x)$, $f''(x)$ and $f''(\frac{\pi}{2})$

$$\begin{aligned} \text{solution: } f'(x) &= (2\sin x - x \cdot \cos x)' \\ &= (2\sin x)' - (x \cdot \cos x)' \\ &= 2(\sin x)' - (x \cdot \cos x)' \\ &= 2\cos x - (\cos x + x \cdot (-\sin x)) \\ &= \boxed{2\cos x - \cos x + x \cdot \sin x} = \boxed{\cos x + x \cdot \sin x} \end{aligned}$$

$$(\sin x)' = \cos x$$

$$\begin{aligned} (x \cdot \cos x)' &\stackrel{\text{Product Rule}}{=} x' \cdot \cos x + x \cdot (\cos x)' \\ &= 1 \cdot \cos x + x \cdot (-\sin x) \end{aligned}$$

$$\begin{aligned} f''(x) &= (f'(x))' = (\cos x + x \cdot \sin x)' \\ &= (\cos x)' + (x \cdot \sin x)' \\ &= -\sin x + x' \cdot \sin x + x \cdot (\sin x)' \\ &= -\sin x + 1 \cdot \sin x + x \cdot \cos x = \boxed{x \cdot \cos x} \end{aligned}$$

$$f''(\frac{\pi}{2}) = \frac{\pi}{2} \cdot \cos \frac{\pi}{2} = \frac{\pi}{2} \cdot 0 = \boxed{0}$$

eg2. Let $h(\theta) = \frac{2\theta}{\sec \theta}$. Find $h'(\theta)$ and $h'(\frac{\pi}{4})$

$$h'(\theta) = \frac{(2\theta)' \cdot \sec \theta - (2\theta) \cdot (\sec \theta)'}{(\sec \theta)^2} = \frac{2 \cdot \sec \theta - 2\theta \cdot \tan \theta \sec \theta}{(\sec \theta)^2}$$

$$\begin{aligned} \text{Hint: } (2\theta)' &= 2 \\ (\sec \theta)' &= \tan \theta \sec \theta \end{aligned}$$

$$= \frac{2\sec \theta (1 - \theta \cdot \tan \theta)}{(\sec \theta)^2}$$

$$= \frac{2(1 - \theta \cdot \tan \theta)}{\sec \theta}$$

$$h'(\frac{\pi}{4}) = \frac{2(1 - \frac{\pi}{4} \cdot \tan \frac{\pi}{4})}{\sec \frac{\pi}{4}} = \boxed{\frac{2(1 - \frac{\pi}{4} \cdot 1)}{\sqrt{2}}}$$

$$\text{Hint: } \tan \frac{\pi}{4} = 1$$

$$\sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \sqrt{2}$$

eg3. Let $y(x) = \csc(x)$. Find $\frac{dy}{dx} \Big|_{x=\frac{\pi}{6}}$ and the tangent line at $x = \frac{\pi}{6}$.

Rank: $(\csc x)' = -\csc x \cdot \cot x$ is in the formula sheet. (Not required).

$$\frac{dy}{dx} = -\csc x \cdot \cot x, \quad \frac{dy}{dx} \Big|_{x=\frac{\pi}{6}} = -\csc \frac{\pi}{6} \cdot \cot \frac{\pi}{6} = -2 \cdot \sqrt{3}$$

Tangent line: slope = $-2\sqrt{3}$, Point: $(\frac{\pi}{6}, \csc \frac{\pi}{6}) = (\frac{\pi}{6}, 2)$

$$\boxed{y - 2 = -2\sqrt{3} \cdot (x - \frac{\pi}{6})}$$

• Application of the formula: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

eg4. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(3x)}{2x}$. Rank: It is $\frac{0}{0}$ type if you plug in 0.

$$\begin{aligned} \text{solution: } \lim_{x \rightarrow 0} \frac{\sin(3x)}{2x} &= \lim_{x \rightarrow 0} \frac{\sin(3x)}{2x} \cdot \frac{3x}{3x} = \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \cdot \frac{3x}{2x} \\ &= \left[\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \right] \cdot \frac{3}{2} \\ &= 1 \cdot \frac{3}{2} = \boxed{\frac{3}{2}} \end{aligned}$$

★ eg5. Find the limit $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2-1}$

$$\begin{aligned} \text{Solution: } \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2-1} &= \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x-1} \cdot \frac{x-1}{x^2-1} \\ &= \left[\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x-1} \right] \cdot \left[\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} \right] \\ &= 1 \cdot \lim_{x \rightarrow 1} \frac{x-1}{(x+1)(x-1)} \\ &= 1 \cdot \frac{1}{1+1} = \boxed{\frac{1}{2}} \end{aligned}$$